Definitions and key facts for section 3.1

Deriving the definition of the 3×3 determinant requires algebra of the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}) \end{bmatrix}.$$

Notice the (3,3) entry is the product of a_{11} and $\det A$.

Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be an $n \times n$ matrix. We define the **determinant of** A, written $\det A$ or $|A|$, as follows:

$n = 1$: $\det A = \det \begin{bmatrix} a \end{bmatrix} = a$.

$n \geqslant 2$: Define A_{ij} to be the $(n-1) \times (n-1)$ matrix obtained by *deleting* the ith row and jth column of A, then

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

Given $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, the (i,j)**- cofactor** of A is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

The cofactor expansion across the ith row of A is

$$\sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

while the cofactor expansion down the jth column of A is

$$\sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Fact: The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

Fact: If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.